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of the equation; for otherwise some line through the origin would cut the curve in five points, which is impossible.

We may also apply certain of the foregoing results to sextic curves whose equation is of the form

$$a_0x^6+a_1x^4y^2+a_2x^2y^4+a_3y^6+a_4x^4+a_5x^2y^2$$
$$+a_6y^4+a_7x^2+a_8y^2+a_9=0, \quad a_9\neq 0.$$

Since this curve does not pass through the origin, its double points, points of inflection, and cusps can enter only by fours. Then from (1) it follows that $\rho=0$, 4, or 8; and $\delta=0$, 4, 8, or 12. Also, from (2) $m \not< 3$. Therefore, equations (1) to (4) yield only the following sets of values:

	I.	II.	III.	IV.	V.	VI.	VII.
$\rho=$	0,	0,	0,	0,	4,	4,	8.
$\delta=$	0,	4,	8,	12,	0,	4,	0.
$\iota=$	72,	48,	24,	0,	40,	16,	8.
$\tau=$	324,	156,	52,	12,	40,	36,	0.
$m=$	30,	22,	14,	6,	18,	10,	6.

ON THE DETERMINATION OF CONICS THROUGH TWO POINTS, THE MAJOR AXIS AND ONE FOCUS BEING GIVEN.

By T. H. HILDEBRANDT, The University of Chicago.

In the consideration of the question of the determination of constants in the application of the Principle of Least Action to Planetary Motion, we need to solve the following problem:

"Given two points, a focus, and a major axis, of a conic, besides its nature (*i. e.*, ellipse, hyperbola, or parabola). Required the number of conics that will satisfy the given conditions." This problem has been solved for the case of the ellipse by Jacobi.[*] The regions in which the second point must lie in order that we obtain a real solution, have also been determined.[†] So far as I know, however, the problem has not been considered for the cases in which the conic is an hyperbola or a parabola.

I. *The Hyperbola.* We need to use the following property of the hyperbola: The difference of the lengths of the focal radii of an hyperbola is equal in length to the major axis.

Let S be one focus and F another, and let P be a point on the hyperbola with distance $SP=r$. Let $2a$ be the length of the major axis. Then if P lies on the branch of the hyperbola adjacent S, we have

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[*] Jacobi, *Vorlesungen ueber die Dynamik.* Werke, Suppl., p. 48.

[†] Todhunter, *Researches in Calculus of Variations,* p. 162.

$$PF=r+2a.$$

If P lies on the branch adjacent F, then

$$PF=r-2a.$$

Suppose S is the given focus, P and Q the given points. Let $PS=r_0$, $QS=r_1$, and let $2a$ be the length of the major axis. Denote by F the second focus of the hyperbola. Then we must distinguish between the following four possibilities:

I. P and Q both on the branch of the hyperbola adjacent S.
II. P and Q both on the branch of the hyperbola adjacent F.
III. P on the branch adjacent S and Q on the branch adjacent F.
IV. P on the branch adjacent F and Q on the branch adjacent S.

Case I. (See Fig. I) P and Q on the Branch Adjacent S.

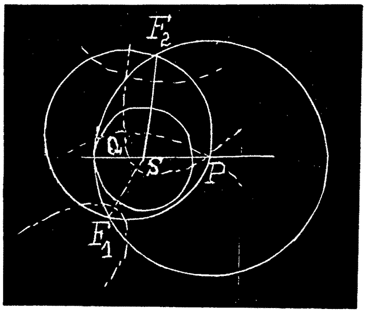

Fig. I.

From the above it is evident that the second foci of the hyperbolae through P will lie on the circle of radius $2a+r_0$ with P as center. The second foci of the hyperbolae through Q will lie on the circle of radius $2a+r_1$ with Q as center. Hence the second foci of the hyperbolae through P and Q will be the intersections of these two circles. There will always be two real intersections, but they may be identical. Let us consider when they coincide, that is, when the circles are tangent to each other. Evidently the circles will be tangent internally. Hence the distance PQ will be the difference of their radii:

$$r_0-r_1 \text{ or } r_1-r_0 \text{ according as } r_1<>r_0.$$

The locus of the points Q for which this happens is then the half ray SP. Evidently when Q lies on this line the hyperbolae degenerate into the straight line SPQ. Since r_0-r_1 is the smallest distance possible between two points P and Q we see that there will be no points Q for which the two circles will not intersect. We have then: through any two points P and Q it is possible to pass two hyperbolae P and Q lying on the branch adjacent S, provided P, Q, and S do not lie on the same straight line, with P and Q on the same side of S.

Case II. (See Fig. II) P and Q on the Branch Adjacent F.

In this case the second foci are the intersections of the circle of radius r_0-2a and center P with the circle of radius r_1-2a and center Q. Evidently we must have $r_0>2a$ and $r_1>2a$, *i. e.*, P and Q lying outside the circle of radius $2a$ with S as center. Moreover, with P fixed it is easily evident that there are points Q for which the circles do not intersect or are tangent Let us find the locus of the points Q for which they are tangent. The circles will be tangent externally, and so we have

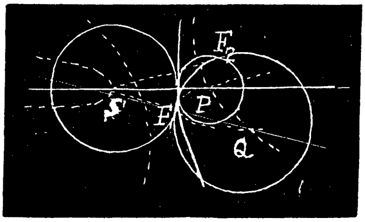

Fig. II.

$$PQ=(r_0-2a)+(r_1-2a)=r_0+r_1-4a.$$

Moreover, $QS=r_1$. Hence, $PQ-QS=r_0-4a$ or $QS-PQ=4a-r_0$, according as $r_0> <4a$.

Therefore these points Q satisfying the condition $PQ-QS=r_0-4a=$a constant, trace a branch of an hyperbola with foci S and P. This hyperbola passes through O, the point of intersection of SP with the circle around P. If $r_0<4a$, the branch will be adjacent to P, and if $r_0>4a$ it will be adjacent S. In either case, if Q lies to the right of this branch, there are no real hyperbolae satisfying the given conditions. If Q lies to the left, there are two real hyperbolae and when Q lies on the branch of the hyperbola, the two required hyperbolae coincide.

Cases III and IV are really not of interest from the standpoint of the Calculus of Variations. We discuss them to make the discussion complete and on account of their intrinsic interest.

Case III. (See Fig. III) P on the Branch Adjacent S and Q on the Branch Adjacent F.

The second foci are the intersections of the circle of radius $2a+r_0$, center P, with the circle of radius r_1-2a, center Q. We must evidently have $r_1>2a$ in order to obtain real solutions. Moreover, there exist points Q for which the two circles do not intersect or are tangent. Let us find the locus of the points Q for which they are tangent. They will be tangent externally. Hence

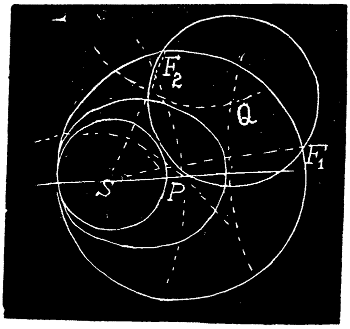

Fig. III.

$$PQ=2a+r_0-(r_1-2a)=4a+r_0-r_1,\quad QS=r_1,$$

and therefore $PQ+QS=4a+r_0$; *i. e.*, these points Q trace an ellipse with foci

P and S and major axis $4a+r_0$. For points Q within this ellipse no real solutions are possible; for points without, we have two real hyperbolae, and for points on the curve, the hyperbolae are coincident.

Case IV. (See Fig. IV) P on the Branch Adjacent F and Q on the Branch Adjacent S.

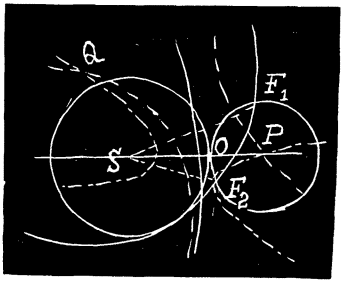

Fig. IV.

In this case the second foci of the hyperbolae through P and Q are the intersections of the circle of radius r_0-2a, center P, with the circle of radius r_1+2a, center Q. It is easily seen that there are points Q for which the two circles do not intersect or are tangent. We find the locus of the latter points. The circles will evidently be tangent internally. Hence,

$$PQ=2a+r_1-(r_0-2a)=4a+r_1-r_0,\quad QS=r_1,$$

and therefore $PQ-QS=4a-r_0$ or $QS-PQ=r_0-4a$, according as $r_0><4a$.

Hence the points Q trace a branch of an hyperbola with P and S as foci and major axis $4a-r_0$ or r_0-4a. This is the second branch of the hyperbola of Case II. It will be adjacent S if $r_0<4a$ and adjacent P if $r_0>4a$. For all points Q lying to the left of this curve there are no real hyperbolae satisfying the given conditions; for points to the right there are two; and for points on the curve, the hyperbolae become coincident.

Figs. V and VI show a comparison of the regions within which Q must lie in order that it be possible to draw real hyperbolae through P and Q satisfying the respective conditions.

It is easily evident that the ellipse of Case IV is the envelope of the hyperbolae through a point P lying on the branch of the hyperbolae adjacent the given focus S. The hyperbola of Cases II and IV is the envelope of the hyperbolae through a point P lying on the branch of the hyperbolae remote from the focus S.

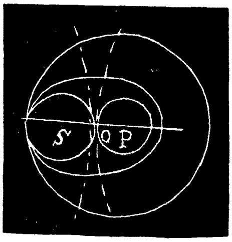

Fig. V.

Case II. Points to the right of - - - -
Case III. Points outside ellipse.
Case IV. Points to the left of — - — - —

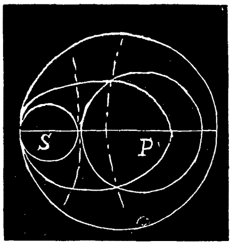

Fig. VI.

Case II. Points to the right of - - - -
Case III. Points outside ellipse.
Case IV. Points to the left of —·- — - —

Parabolae. (See Fig. VII). In order to find the parabolae through two given points P and Q, having a given focus S, we need to find the directrices. The directrices of the parabolae through P are evidently the tangents to the circle of radius SP and center P, and those of the parabolae through Q, the tangents to the circle of radius SQ and center Q. Hence the parabolae through P and Q have the common tangents of these two circles as directrices. These tangents will always be real, and distinct, unless P, Q, and S lie on the same straight line and P and Q lie on the same side of S. Then there is only one common tangent, one that passes through S, and hence in this case the parabolae become coincident and degenerate into the straight line PQS. There is only one other exceptional case, when P, Q, and S lie on the same line and P and Q lie on opposite sides of S. Then there will be three common tangents, and there will consequently be three real parabolae through P and Q. One of these is the straight line PQS.

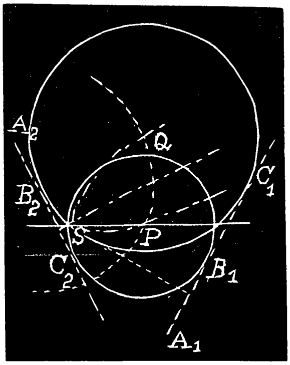

Fig. VII.

The parabolae through a point P with focus S have no envelope.

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DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

295. Proposed by CHARLES GILPIN, JR., Philadelphia, Pa.

In the equation $x^3-ax\pm b=0$, we have the following relation between the coefficients and the roots: (1) When $a^3/b^2=6.75$ there are three real roots, two of which are equal; (2) when $a^3/b^2<6.75$ there are two imaginary roots and one real one; and (3) when $a^3/b^2>6.75$ there are three real, unequal roots.

Remarks and Solution by G. B. M. ZERB, A. M., Ph. D., Philadelphia, Pa., and H. V. SPUNAR, East Pittsburg, Pa.

The proposer has interchanged cases (2) and (3). When $a^3/b^2<6.75$, there are three real unequal roots; when $a^3/b^2>6.75$ there are two imaginary and one real root.

Let $m^3=\pm\frac{1}{2}b+\sqrt{\frac{b^2}{4}-\frac{a^3}{27}}$, $n^3=\pm\frac{1}{2}b-\sqrt{\frac{b^2}{4}-\frac{a^3}{27}}$, and ω and ω^2=the